Continue on Fourier Series and PDE

Consider $f \in R[0, 2L]$, $L > 0$

A change of variable:

$$g(x) := f\left(\frac{L}{\pi} x\right)$$

Note $g \in R[0, 2\pi]$.

Then we can compute Series of g, and change the variable back to obtain Fourier Series of f on $[0, 2L]$.

$$g(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx$$

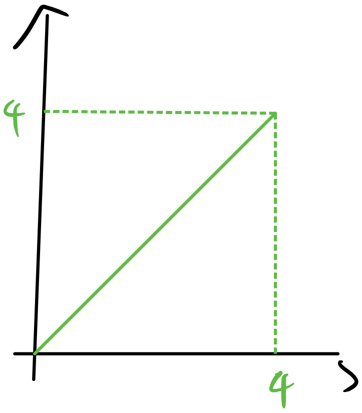
$$f(y) = g\left(\frac{\pi}{L} y\right)$$

$$\sim A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L} y\right) + B_n \sin\left(\frac{n\pi}{L} y\right)$$

Example 1:

$$f : [0, 4] \to \mathbb{R}$$

$$f(x) = x.$$



Change Variable:

$$g(x) = f\left(\frac{2}{\pi}x\right) = \frac{2}{\pi}x$$

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{2}{\pi}x \, dx = \frac{1}{\pi^2} \cdot \frac{1}{2}x^2 \Big|_0^{2\pi} = 2$$

$$A_n = \frac{1}{\pi}\int_0^{2\pi} \frac{2}{\pi}x \cos nx \, dx$$

$$= \frac{2}{\pi^2}\left[\frac{1}{n} x \sin nx \Big|_0^{2\pi} - \frac{1}{n}\int_0^{2\pi} \sin nx \, dx \right]$$

$$= 0$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} \frac{2}{\pi}x \sin nx \, dx$$

$$= \frac{2}{\pi^2}\left[-\frac{1}{n} x \cos nx \Big|_0^{2\pi} + \frac{1}{n}\int_0^{2\pi} \cos nx \, dx \right]$$

$$= -\frac{4}{n\pi}$$

$$g(x) \sim 2 - \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin nx$$

$$f(x) \sim 2 - \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin\left(\frac{\pi n}{2} x\right)$$

Recall $g(x) = f(\frac{L}{\pi} x)$

Then $g(\frac{\pi}{L} y) = f(y)$

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} g(x)\, dx$$

$$\overset{x \mapsto \frac{\pi}{L} y}{=} \frac{1}{2\pi} \int_0^{2\pi} g(\tfrac{\pi}{L} y)\, d(\tfrac{\pi}{L} y)$$

$$= \frac{1}{2L} \int_0^{2L} f(y)\, dy$$

$$A_n = \frac{1}{\pi} \int_0^{2\pi} g(x) \cos nx\, dx$$

$$\overset{x \mapsto \frac{\pi}{L} y}{=} \frac{1}{\pi} \int_0^{2\pi} g(\tfrac{\pi}{L} y) \cos n(\tfrac{\pi}{L} y)\, d(\tfrac{\pi}{L} y)$$

$$= \frac{1}{L} \int_0^{2L} f(y) \cos(\tfrac{n\pi}{L} y)\, dy$$

$$B_n = \frac{1}{\pi} \int_0^{2\pi} g(x) \sin nx\, dx$$

$$\overset{x \mapsto \frac{\pi}{L} y}{=} \frac{1}{\pi} \int_0^{2\pi} g(\tfrac{\pi}{L} y) \sin n(\tfrac{\pi}{L} y)\, d(\tfrac{\pi}{L} y)$$

$$= \frac{1}{L} \int_0^{2L} f(x) \sin(\tfrac{\pi}{L} y)\, dy$$

$\Rightarrow$ We can directly compute Fourier Series on $[0, 2L]$.

Example 2:

$$(1)\quad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}, & (x,t) \in (0,\pi) \times (0,\infty) \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(\pi,t) = 0, & t \in (0,\infty) \\ u(x,0) = x^2,\ \dfrac{\partial u}{\partial t}(x,0) = 0, & x \in [0,\pi]. \end{cases}$$

Even Extension

$$v : [-\pi, \pi] \times [0,\infty) \to \mathbb{R},$$

$$v(x,t) = \begin{cases} u(x,t), & \text{if } x \in [0,\pi] \\ u(-x,t), & \text{else.} \end{cases}$$

Initial Condition:

For $x < 0$,

$$v(x,0) = u(-x,0) = x^2.$$

$$\frac{\partial v}{\partial t}(x,0) = -\frac{\partial u}{\partial t}(-x,0) = 0$$

The Problem becomes:

$$(2)\quad \begin{cases} \dfrac{\partial^2 v}{\partial t^2} = \dfrac{\partial^2 v}{\partial x^2}, & (x,t) \in (-\pi,\pi) \times (0,\infty) \\ \dfrac{\partial v}{\partial x}(-\pi,t) = \dfrac{\partial v}{\partial x}(\pi,t) = 0, & t \in (0,\infty) \\ v(x,0) = x^2,\ \dfrac{\partial v}{\partial t}(x,0) = 0, & x \in [-\pi,\pi]. \end{cases}$$

To make sense of the extension,

Consider the New Problem (2) first,

and forget about the original Problem.

$$(2)\ -\ \begin{cases} \dfrac{\partial^2 v}{\partial t^2} = \dfrac{\partial^2 v}{\partial x^2}, & (x,t) \in (-\pi, \pi) \times (0, \infty) \\ \dfrac{\partial v}{\partial x}(-\pi, t) = \dfrac{\partial v}{\partial x}(\pi, t) = 0, & t \in (0, \infty) \\ v(x,0) = x^2,\ \dfrac{\partial v}{\partial t}(x,0) = 0, & x \in [-\pi, \pi]. \end{cases}$$

The Problem has unique solution

(To see this, consider $w_1 = v_1 - v_2$,

v_1, v_2 solutions of (2).)

And the solution of the problem is even.

(To see this, consider $v_2(x,t) = v_1(-x,t)$

where $v(x)$ is a solution of (1)

and by uniqueness of solution, $v(x,t) = v(-x,t)$)

Finally, the restriction $u = v\big|_{[0,\pi] \times [0,\infty)}$

solve the original problem (1).

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} = \dfrac{\partial^2 v}{\partial x^2}, & (x,t) \in (-\pi, \pi) \times (0, \infty) \\ \dfrac{\partial v}{\partial x}(-\pi, t) = \dfrac{\partial v}{\partial x}(\pi, t) = 0, & t \in (0, \infty) \\ v(x,0) = x^2, \ \dfrac{\partial v}{\partial t}(x,0) = 0, & x \in [-\pi, \pi]. \end{cases}$$

Let $v(x,t) = X(x)\,T(t)$.

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2} \Rightarrow \frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \lambda$$

$$X''(x) = \lambda X(x)$$

$$\Rightarrow X(x) = \begin{cases} \alpha_1 e^{\sqrt{\lambda} x} + \alpha_2 e^{-\sqrt{\lambda} x}, & \lambda > 0 \\ \alpha_1 x + \alpha_2, & \lambda = 0 \\ \alpha_1 \cos\sqrt{-\lambda} x + \alpha_2 \sin\sqrt{-\lambda} x, & \lambda < 0 \end{cases}$$

If $\lambda > 0$, BC $\Rightarrow \alpha_1 = \alpha_2 = 0$

If $\lambda = 0$, BC $\Rightarrow \alpha_1 = 0$

$$T''(t) = \lambda T(t)$$

$$\Rightarrow T(t) = \beta_1 t + \beta_2$$

If $\lambda < 0$, BC $\Rightarrow \sqrt{-\lambda} = n \in \mathbb{N}$,

$$\alpha_2 = 0$$

$$T''(t) = \lambda T(t)$$

$$\Rightarrow T(t) = \beta_1 \cos nt + \beta_2 \sin nt.$$

$$v_0(x,t) = A_0' t + A_0$$

$$v_n(x,t) = (A_n \cos nt + B_n \sin nt) \cos nx$$

$$v(x,t) = \sum_{n=0}^{\infty} v_n(x,t).$$

$$= A_0' t + A_0 + \sum_{n=1}^{\infty} (A_n \cos nt + B_n \sin nt) \cos nx$$

Fourier Series of x^2 : $\frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$

$\therefore A_0 = \frac{\pi^2}{3}$, $A_n = \frac{4}{n^2}(-1)^n$, $B_n = 0$

$\frac{\partial v}{\partial t}(x,0) = 0 \Rightarrow A_0' = 0.$

$$\therefore u(x,t) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nt \cos nx$$

Discrete Fourier Transform

$$\vec{f} = \begin{bmatrix} f_0 \\ \vdots \\ f_{n-1} \end{bmatrix} \in \mathbb{C}^n : \text{ Measurement / Data of function } f$$

The DFT:

$$\vec{c} = \begin{bmatrix} c_0 \\ \vdots \\ c_{n-1} \end{bmatrix} \in \mathbb{C}^n \text{ where } c_k = \frac{1}{n} \sum_{j=0}^{n-1} f_j \, e^{-i\left(\frac{2jk\pi}{n}\right)}$$

Or write : $(A_w)_{j,k} = w^{(j-1)(k-1)}$

$$A_w = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{n-1} \\ 1 & w^2 & w^4 & & w^{2(n-1)} \\ \vdots & \vdots & & & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{(n-1)(n-1)} \end{bmatrix},$$

$$w = e^{2\pi i/n}$$

$$\vec{c} = \frac{1}{n} \overline{A_w} \vec{f}$$

What Discrete Fourier Transform do is also a change of basis

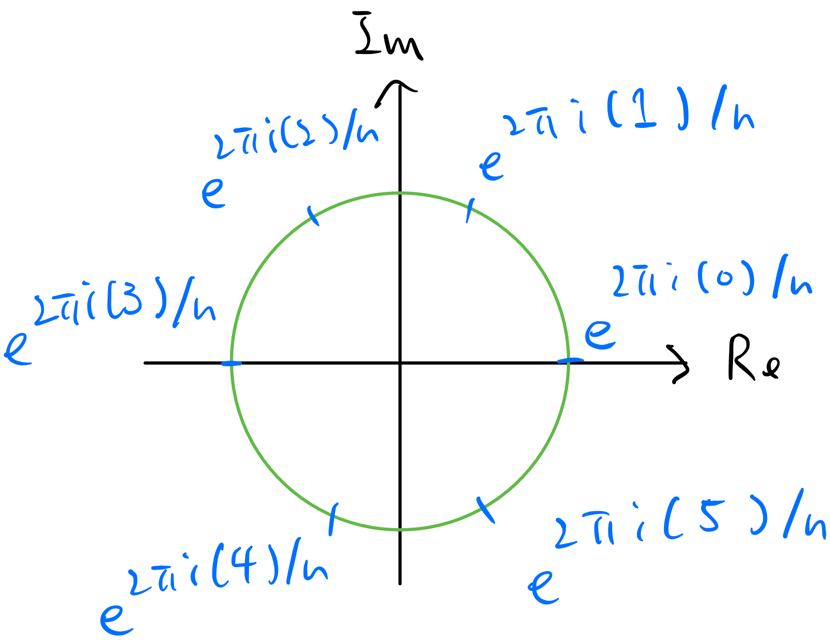
$$\left(\vec{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \rightarrow \vec{u}_j = \begin{bmatrix} w^{(j-1)(0)} \\ w^{(j-1)(1)} \\ \vdots \\ w^{(j-1)(n-1)} \end{bmatrix} \right)$$

Note $\vec{u}_j \cdot \vec{u}_k = \sum_h e^{2\pi i(j-1)(h-1)/n} \cdot \overline{e^{2\pi i(k-1)(h-1)/n}}$

$$= \sum_h e^{2\pi i(j-k)(h-1)/n}$$

$$= \begin{cases} n, & \text{if } j=k \\ 0, & \text{else.} \end{cases}$$

Graphical Illustration for $n=6$:



$A_\omega \cdot A_\omega^* = n I$ $\qquad A_\omega^* = \overline{A_\omega^T}$

(Conjugate Transpose)

$\Rightarrow \left(\frac{1}{\sqrt{n}} A_\omega\right)\left(\frac{1}{\sqrt{n}} A_\omega^*\right) = I$

$$\therefore \left\{ \frac{1}{\sqrt{n}} \vec{u}_j \right\}_{j=1}^{n} = \left\{ \frac{1}{\sqrt{n}} \begin{bmatrix} e^{2\pi i (j-1)(0)/n} \\ e^{2\pi i (j-1)(1)/n} \\ \vdots \\ e^{2\pi i (j-1)(n-1)/n} \end{bmatrix} \right\}_{j=1}^{n}$$

is an orthonormal basis.

For simpler calculation,

Take $\frac{1}{n}\overline{A_\omega}$ as Forward Transform,

A_ω as Backward Transform.

Inverse Discrete Fourier Transform:

$\vec{f} = A_\omega \vec{c}$.

Exercise

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t), & (x,t) \in (0,\pi) \times (0,\infty) \\ u(0,t) = u(\pi,t) = 0, & t \in [0,\infty) \\ u(x,0) = x(x-\pi), & x \in [0,\pi]. \end{cases}$$

Consider an odd extension:

$$v(x) = \begin{cases} u(x,t), & x \geq 0 \\ -u(-x,t), & x < 0 \end{cases}$$

Exercise Solution

$$\begin{cases} \dfrac{\partial v}{\partial t}(x,t) = \dfrac{\partial^2 v}{\partial x^2}(x,t), \quad (x,t) \in (-\pi, \pi) \times (0, \infty) \\ v(-\pi, t) = v(\pi, t) = 0, \quad t \in [0, \infty) \\ v(x, 0) = \begin{cases} x(x-\pi), & x \geq 0 \\ -x(x+\pi), & x < 0 \end{cases} \end{cases}$$

$$v(x,t) = X(x)\,T(t)$$

$$\frac{T'}{T} = \frac{X''}{X} = \lambda$$

$$X(x) = \begin{cases} \alpha_1 e^{\sqrt{\lambda} x} + \alpha_2 e^{-\sqrt{\lambda} x}, & \lambda > 0 \\ \alpha_1 x + \alpha_2, & \lambda = 0 \\ \alpha_1 \cos\sqrt{-\lambda} x + \alpha_2 \sin\sqrt{-\lambda} x, & \lambda < 0 \end{cases}$$

$\lambda > 0$, BC $\Rightarrow$ $\alpha_1 = \alpha_2 = 0$

$\lambda = 0$, BC $\Rightarrow$ $\alpha_1 = 0$, $T(t) = \beta$

$\lambda < 0$, BC $\Rightarrow$ $\sqrt{-\lambda} = n \in \mathbb{N}$, $\alpha_2 = 0$,
$T(t) = \beta e^{-n^2 t}$

$$V_0 = A_0$$

$$V_n = B_n \sin nx \, e^{-n^2 t}$$

$$v(x,t) = A_0 + \sum_{n=1}^{\infty} B_n \sin nx \, e^{-n^2 t}$$

$$v(x,0) = \begin{cases} x(x-\pi), & x \geq 0 \\ -x(x+\pi), & x < 0, \end{cases}$$

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} v(x,0) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x(x-\pi) \sin nx \, dx$$

$$= -\frac{2}{\pi n} \int_0^{\pi} x(x-\pi) \, d(\cos nx)$$

$$= -\frac{2}{\pi n} \left[x(x-\pi)\cos nx \Big|_0^{\pi} - \int_0^{\pi} \cos nx \, d(x(x-\pi)) \right]$$

$$= \frac{2}{\pi n} \int_0^{\pi} \cos nx \, (2x) \, dx - \frac{2}{n} \int_0^{\pi} \cos nx \, dx$$

$$= \frac{4}{\pi n^2} \int_0^{\pi} x \, d(\sin nx) - \frac{2}{n^2} \sin nx \Big|_0^{\pi}$$

$$= \frac{4}{\pi n^2} \left[x \sin nx \Big|_0^{\pi} - \int_0^{\pi} \sin nx \, dx \right]$$

$$= \frac{4}{\pi n^3} \cos nx \Big|_0^{\pi} = \frac{4}{\pi n^3} ((-1)^n - 1)$$

$$\therefore \; A_0 = 0, \; B_n = \frac{4}{\pi n^3} ((-1)^n - 1)$$

$$u(x,t) = v(x,t), \; (x,t) \in [0,\pi] \times [0,\infty)$$